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Heat transfer from turbulent separated flows

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A power-law relation is derived between the Stanton number and the Reynolds number, expressing the law of heat transfer for a wall adjacent to a region of turbulent separated flow. The derivation is based on Prandtl's (1945) proposal for the laws of dissipation, diffusion and generation of turbulent kinetic energy. The constants appearing in these laws are determined by reference to experimental data for the hydrodynamic properties of the constant-stress and the linear-stress layers.

The agreement between the resulting predictions and the experimental data of other workers is sufficiently good to suggest that the actual mechanism of heat transfer from separated flows has much in common with that which is postulated. Closer agreement can be expected only after the present one-dimensional analysis has been superseded by a two-dimensional one.

1. Introduction

1.1. *The problem considered*

Turbulent separated flows occur, at sufficiently high Reynolds numbers, at the rear of bluff bodies, in ducts downstream of abrupt enlargements, and in boundary layers upstream or downstream of step-like discontinuities in wall profile. They are accidental features of some classes of engineering equipment; and in others they may be deliberately introduced in order to augment heat-transfer rates to nearby walls. Current interest in flows of this type is attested by many recent publications; see for example Hansen 1964.

The laws of heat transfer obeyed by turbulent separated flows exhibit characteristic features which have escaped explanation. One of these, to which Hanson & Richardson (1964), Richardson (1963), and Sogin (1964) have drawn special attention, is that the Stanton number is usually proportional to the $-\frac{1}{3}$ power of the Reynolds number; this dependence contrasts strongly with that appropriate to attached turbulent boundary layers, for which the exponent is close to $-0{\cdot}2$; it differs also from that of laminar boundary layers, which exhibit a square-root dependence. Another typical feature is that the local value of the heat flux depends almost entirely on the difference between the temperature of the stream and the local temperature of the wall; the temperature of the wall at nearby points has little influence. This behaviour, which simplifies the task of the designer, contrasts strongly with that exhibited by attached boundary layers, whether laminar or turbulent.

Of particular interest is that the heat flux often exhibits a maximum value at the location of boundary-layer re-attachment; values of heat flux three or four times as great as those prevailing a short distance downstream have been reported by Seban, Emery & Levy (1959) and by Ede, Hislop & Morris (1956). This fact assumes particular interest when it is recalled that, at the re-attachment point, the (time–mean) shear stress is zero; and most formulae for calculating heat-transfer rates which are based on a physical theory of flow in the boundary layer would predict that, where the shear stress is zero, the heat transfer rate must also vanish.

To bring understanding of heat transfer from separated flows up to the modest level of satisfactoriness appertaining to heat transfer from attached boundary layers, a theory is required which is based on different assumptions from those usually employed. Such a theory is provided by the present paper.

1.2. *Outline of the present contribution*

One of the characteristic features of separated flows is that the locations of maximum shear stress are remote from the wall; indeed the greatest stresses are commonly to be found in layers which are similar, particularly in respect of lack of influence of viscosity, to the free-mixing layer studied by Liepmann & Laufer (1947). This is a second point of contrast with the attached boundary layer on a flat plate, for example; for, in such a boundary layer, the maximum shear stress occurs at the wall itself, where of course the viscosity exerts its maximum influence.

A consequence of this feature is that the turbulence which is generated in the remote high-shear region of a separated flow must be conveyed to the vicinity of the wall by the action of convection and diffusion; the turbulence intensity in the vicinity of the wall, which is a main determinant of heat transfer, is governed by the interaction of these two factors with turbulence dissipation. By contrast, the turbulence level near a wall having an attached boundary layer is governed by the balance between the generation of turbulence near the wall and the dissipation which occurs in the same locality. This, at any rate, is the view of the situation which is adopted in the present paper.

To convert these qualitative notions into a quantitative theory, it is necessary to postulate mathematical relationships describing the processes of generation, convection, diffusion and dissipation of turbulent energy. Fortunately, several authors have made proposals which may be utilized, including Prandtl (1945), Nevzglyadov (1945), Emmons (1954), Townsend (1961), and Glushko (1965); the present writer's interest was stimulated particularly by the latter paper, but the ideas used are already quite explicit in Prandtl's paper. None of the authors has applied the equations to separated flows.

All separated flows of practical importance are two-dimensional in character; despite this, the present paper concerns a one-dimensional model of the flow near the wall. It must thus be regarded as bearing the same relationship to a complete theory as does a Couette-flow analysis to a complete two-dimensional theory of flow in an attached boundary layer.

In order to determine the constants which appear in the postulated relationships for the generation, diffusion and dissipation of turbulent energy, it is necessary to appeal to experimental information. When doing so, it is convenient to re-examine the constant-shear and linear-shear layers which have already been treated, by Townsend (1961), from a point of view which exhibits both similarities to and differences from that of the present paper. With the constants so determined, a Reynolds-number dependence is predicted for heat transfer which is close to that which is found experimentally. Although the one-dimensionality of the model prevents an absolute prediction of heat-transfer rates, the order-of-magnitude agreement with experimental values is wholly satisfactory.

2. Analysis

2.1. *Definitions and assumptions*

We define the turbulent kinetic energy, k, by way of the equation

$$k \equiv \tfrac{1}{2}\{\overline{(u')^2}+\overline{(v')^2}+\overline{(w')^2}\}. \tag{2.1}$$

Here the quantities u', v' and w' represent the fluctuating components of velocity in the three co-ordinate directions, and the bars denote mean values with respect to time. The quantity k is supposed, like all other dependent variables in our one-dimensional model, to be a function only of the distance, y, from the wall bounding the fluid. We shall suppose that the state of the turbulence of the fluid at a particular point is characterized by only two quantities, namely k and y, the first giving the intensity and the latter the length scale of the turbulence.

The composition of the fluid is supposed to be uniform, and the temperature differences to be small enough to have no influence on the other physical properties of the fluid.

It will be supposed that the intensity of turbulence is zero at the wall ($y = 0$), and that, where y exceeds a definite value, say y_0, the effects of viscous action are negligible. Thus we postulate something akin to a laminar sublayer ($y < y_0$), succeeded by a fully turbulent region ($y > y_0$).

For the fully turbulent region, we postulate the following laws to describe the processes of, respectively: turbulent-energy dissipation, turbulent-energy diffusion and turbulent-energy generation:

$$\text{rate of dissipation per unit volume} = a\rho k^{\frac{3}{2}}/y; \tag{2.2}$$

$$\text{rate of diffusion into unit volume} = b\rho \frac{d}{dy}\left(k^{\frac{1}{2}}y\frac{dk}{dy}\right); \tag{2.3}$$

$$\text{rate of generation per unit volume} = \tau(du/dy), \tag{2.4}$$

where
$$\tau/\rho = \nu_t(du/dy), \tag{2.5}$$

and
$$\nu_t = ck^{\frac{1}{2}}y. \tag{2.6}$$

Here the quantities a, b and c are supposed to be constants; τ stands for the shear stress, ρ for the fluid density, and u for the mean velocity of the fluid in the x-direction along the wall. The quantity ν_t is the 'total' kinematic viscosity, distinguished from its laminar counterpart, ν without subscript, by satisfying (2.5) throughout the region of interest, whether the flow is laminar or turbulent. All the above relationships (apart from (2.5) which is a definition, and which holds whether y is greater or less than y_0) may be justified by way of dimensional analysis. They have been used by Prandtl (1945), Emmons (1954) and Glushko (1965). The remainder of the analysis appears, however, to be novel.

For the region nearer the wall ($y < y_0$), we make the following assumptions:

$$\text{(i)} \quad y_0 k_0^{\frac{1}{2}}/\nu = Y, \quad \text{a const.}; \tag{2.7}$$

$$\text{(ii)} \quad \frac{y_0}{k_0}\left(\frac{dk}{dy}\right)_0 = Y', \quad \text{a const.}; \tag{2.8}$$

$$\text{(iii)} \quad \nu_t/\nu = \tilde{\nu}\{y/y_0\}; \tag{2.9}$$

$$\text{(iv)} \quad \sigma_t = \sigma_t\{y/y_0, \sigma\}. \tag{2.10}$$

Here k_0 is of course the value of k prevailing where y equals y_0. The quantity σ_t is the 'total Prandtl number', that is to say, the local value of ν_t divided by the local value of the total thermal diffusivity, the latter quantity being the one that makes the Fourier heat-conduction law valid throughout the whole region, whether the fluid is laminar or turbulent. The brackets $\{...\}$ signify 'a function of', and σ without subscript stands for the laminar value of the Prandtl number.

We can now write down an equation which expresses the fact that the turbulent kinetic energy is invariant with time. This may be called the energy-balance equation; it runs

$$a\frac{k^{\frac{3}{2}}}{y} - b\frac{d}{dy}\left(k^{\frac{1}{2}}y\frac{dk}{dy}\right) - \frac{\tau}{\rho}\frac{du}{dy} = 0. \tag{2.11}$$

Together with (2.5) and (2.6), this equation will permit us to calculate the distribution of turbulent kinetic energy in the fully turbulent region. Thereafter, we shall be enabled to calculate the temperature distributions, and so derive an expression for the heat-transfer rate.

2.2 *Solution for the zero-shear layer*

At the point of reattachment of a turbulent boundary layer, the shear stress at the wall is zero, as has already been mentioned; and in all separated flows the local rate of generation of turbulence as a consequence of shear stresses is likely to be small in the vicinity of the wall. We therefore turn first to the extreme case in which the shear stress is zero, so that turbulence generation is absent. (2.11) reduces to

$$\frac{3}{2}\frac{a}{b}k^{\frac{3}{2}} - y\frac{d}{dy}\left(y\frac{dk^{\frac{3}{2}}}{dy}\right) = 0. \tag{2.12}$$

The solution of this equation can be written down at once; it is

$$k^{\frac{3}{2}} = \text{const.}\, y^m + \text{const.}\, y^{-m}, \tag{2.13}$$

where the quantity m is defined by

$$m \equiv (\tfrac{3}{2}a/b)^{\frac{1}{2}}. \tag{2.14}$$

By reference to the boundary condition (2.8), the solution can be written as

$$\left(\frac{k}{k_0}\right)^{\frac{3}{2}} = \left(\frac{1+(3/2m)Y'}{2}\right)\left(\frac{y}{y_0}\right)^m + \frac{(m-\frac{3}{2}Y')}{(m+\frac{3}{2}Y')}\left(\frac{y_0}{y}\right)^m. \tag{2.15}$$

Let us now suppose that, at a distance y_1 from the wall, the turbulent kinetic energy has the value k_1. Further, let us suppose that y_1 is very much greater than y_0. Then we can easily deduce that, if m exceeds unity, which will later be shown to be the case, we can write the relation between the kinetic energies at the inner and outer edges of the fully-turbulent layer as

$$\left(\frac{k_1}{k_0}\right)^{\frac{3}{2}} \approx \frac{(1+(3/2m)Y')}{2}\left(\frac{y_1}{y_0}\right)^m. \tag{2.16}$$

We now choose, for later convenience, to introduce a reference velocity u_G and a reference dimension D. These might represent respectively the velocity of the main stream and the diameter of a bluff body, suspended in it, behind which is formed the separated-flow region which we wish to study. If k_0 is eliminated by reference to (2.7), we deduce

$$\frac{\nu}{u_G y_0} \approx \left(\frac{1+(3/2m)Y'}{2}\right)^{-1/(m+3)} Y^{-3/(m+3)} \left(\frac{k_1^{\frac{1}{2}}}{u_G}\right)^{3/(m+3)} \left(\frac{D}{y_1}\right)^{m/(m+3)} \left(\frac{u_G D}{\nu}\right)^{-m/(m+3)}. \tag{2.17}$$

Equation (2.17) is a major result of the analysis. It will lead, in §2.4, to a power-law relation between the Stanton number and the Reynolds number. The latter quantity can be recognized in the last bracket on the right-hand side; and the term on the left-hand side will be shown to be proportional to the Stanton number. All the other terms can be expected to be constants for a given geometry of flow. First, however, we must determine the value of the quantity m; this, as its definition (2.14) shows, depends on the relative size of the constants a and b, which express respectively the rates of dissipation and diffusion. The next section will be devoted to this determination.

2.3. *The determination of constants*

(i) We first turn our attention to the *constant-stress layer*. The energy-balance equation for the fully-turbulent region, (2.11), now becomes

$$\frac{ak^{\frac{3}{2}}}{y} - b\frac{d}{dy}\left(k^{\frac{1}{2}}y\frac{dk}{dy}\right) - \frac{(\tau_S/\rho)^2}{ck^{\frac{1}{2}}y} = 0. \tag{2.18}$$

Here the velocity gradient has been eliminated by reference to (2.5) and (2.6). τ_S is the shear stress throughout the layer.

The solution is a simple one, namely

$$k = (\tau_S/\rho)/(ac)^{\frac{1}{2}}. \tag{2.19}$$

From this there follows
$$\nu_t = c^{\frac{3}{4}}a^{-\frac{1}{4}}(\tau_S/\rho)^{\frac{1}{2}}y. \tag{2.20}$$

Experimental information, as summarized, for example, by Hinze (1959), shows that the turbulent kinetic energy is indeed uniform in a constant-shear layer, the ratio $k/(\tau_S/\rho)$ being approximately equal to 4·0. The same source of experimental information confirms that the total viscosity is proportional to $(\tau_S/\rho)^{\frac{1}{2}}y$, the proportionality constant being about 0·4. Comparison with (2.19) and (2.20) thus leads to

$$a = 0{\cdot}313, \tag{2.21}$$

and
$$c = 0{\cdot}2. \tag{2.22}$$

(ii) It is possible to determine the diffusion constant, b, by the examination of experimental data of several kinds. We here make use of the data which are available for the velocity distribution near a wall under conditions of adverse pressure gradient and small shear stress at the wall; in these conditions *the shear stress τ is linear in y*, so that it is profitable to make an analysis of the distribution of k in a boundary layer for which τ is proportional to y.

Let

$$\tau/\rho = p'y, \tag{2.23}$$

where p' is a constant. Then the differential equation (2.11) becomes

$$\frac{ak^{\frac{3}{2}}}{y} - b\frac{d}{dy}\left(yk^{\frac{1}{2}}\frac{dk}{dy}\right) - \frac{(p'y)^2}{ck^{\frac{1}{2}}y} = 0. \tag{2.24}$$

The solution of this equation is easily shown to be

$$k = \frac{p'y}{\{c(a-\frac{3}{2}b)\}^{\frac{1}{2}}}. \tag{2.25}$$

Combination of this result with (2.5), (2.6) and (2.23) swiftly leads to an expression for the velocity distribution in the neighbourhood of the wall; it is

$$u = 2\left\{\frac{p'(a-\frac{3}{2}b)^{\frac{1}{2}}}{c^{\frac{3}{2}}}\right\}^{\frac{1}{2}} y^{\frac{1}{2}} + \text{const.} \tag{2.26}$$

A similar result was derived by Townsend (1961), whose expressions for the diffusion rate and shear stress differed, however, significantly† from those of (2.3), (2.5) and (2.6).

Townsend showed, by examination of the experimental data of Schubauer & Klebanoff (1951), that the velocity profile indeed obeyed a law like (2.26) in the neighbourhood of the wall, even though the shear stress at the wall was not precisely zero. Townsend expressed the data in the form

$$u = \frac{2(p')^{\frac{1}{2}}}{K_0} y^{\frac{1}{2}} + \text{const.} \tag{2.27}$$

in which K_0 was found by examination of the experimental data to equal $0{\cdot}48 \mp 0{\cdot}03$. We can therefore deduce, by comparison of (2.26) and (2.27), and by introduction of the values of a and c already established, that the value of b lies within the limits shown in the following table. Corresponding values of m and of $m/(m+3)$ are also included in the table for later convenience; they have been deduced via (2.14).

K_0	b	m	$m/(m+3)$
0·45	0·079	2·43	0·45
0·48	0·108	2·08	0·41
0·51	0·130	1·90	0·368

† In place of (2.5) and (2.6), Townsend took the shear stress to be a universal constant times the local kinetic energy; for the zero-stress layer, this would not be satisfactory. In place of (2.3), Townsend assumed that the rate of diffusion into unit volume was proportional to the first differential coefficient of the $\frac{3}{2}$ power of the kinetic energy.

In the following discussion, the value 0·1173 will be adopted for b, to which corresponds a round value of m, namely 2·0.

(iii) In the following table are summarized, for comparison with the values here adopted, the constants proposed by Wieghardt (1945) in an appendix to Prandtl's paper, and by Glushko (1965). Wieghardt based his dissipation constant on data for the decay of isotropic turbulence, his diffusion constant on data for the turbulent-energy distribution near the centre line of a parallel-sided duct, and his total-viscosity constant on the measured velocity distribution near a wall. Glushko chose his total-viscosity constant by reference to data presented by Hinze (1959); the other two constants were fixed by carrying out a large number of integrations of the two-dimensional turbulent-energy equation, simultaneously with those for momentum and continuity, and selecting those values which gave the best agreement with experimental data for velocity and turbulent-energy distributions.

Author	a	b	c	b/a	b/c
Wieghardt	0·45	0·152	0·224	0·338	0·679
Glushko	0·313	0·08	0·2	0·256	0·4
Present	0·313	0·1173	0·2	0·375	0·587

It should be noted that one implication of (2.26) is that b/a cannot exceed $\frac{2}{3}$; for, if it did, the multiplier of $y^{\frac{1}{2}}$ would be imaginary.

2.4. *Heat transfer through the turbulent layer and laminar sublayer*

(i) Having established the exponents on the right-hand side of (2.17), we now turn our attention to expressing the left-hand side in terms of a dimensionless measure of the heat-transfer coefficient, the Stanton number. If the heat flux through the layer is q, and the temperature measured above that of the wall is t, the Fourier heat-conduction equation can be written as

$$q = c_p \rho \frac{\nu_t}{\sigma_t} \frac{dt}{dy}, \tag{2.28}$$

where c_p stands for the specific heat of the fluid at constant pressure.

Now the Stanton number S is defined by

$$S \equiv q/(t_1 c_p \rho u_G), \tag{2.29}$$

where t_1 is the temperature of the fluid at the outer boundary of the turbulent layer, i.e. in the main stream. We can therefore deduce

$$S = \frac{\nu}{u_G y_0} \left[\int_0^{y_1/y_0} \sigma_t \frac{\nu}{\nu_t} \, d\left(\frac{y}{y_0}\right) \right]^{-1}. \tag{2.30}$$

Thus the Stanton number is proportional to $\nu/(u_G y_0)$, as was stated at the end of §2.2. We shall now derive an expression for the proportionality constant, represented by the contents of the square bracket.

(ii) Let us suppose that, in the fully turbulent region, the total Prandtl number has a constant value; let the symbol for this be σ_{tt}. Following the practice of Spalding & Jayatillaka (1964), we split the integral into two parts, thus

$$\int_0^{y_1/y_0} \sigma_t \frac{\nu}{\nu_t}\, d\left(\frac{y}{y_0}\right) = \sigma_{tt}\left[\int_0^{y_1/y_0} \left(\frac{\sigma_t}{\sigma_{tt}} - 1\right)\frac{\nu}{\nu_t}\, d\left(\frac{y}{y_0}\right) + \int_0^{y_1/y_0} \frac{\nu}{\nu_t}\, d\left(\frac{y}{y_0}\right)\right]. \tag{2.31}$$

Now we can suppose that σ_t differs from σ_{tt} only in the laminar sublayer; the upper integration limit of the first integral can therefore be placed equal to unity. Secondly, because of the hypothesis expressed by (2.9) and (2.10), we can evaluate the integral by reference to the extensive experimental data for the constant-stress layer, for which $(\tau_S/\rho)^{\frac{1}{2}}$ is equal to $0{\cdot}5\,k_0^{\frac{1}{2}}$ as stated in §2.3. Thus the first integral in the square bracket of (2.4) can be expressed as

$$\int_0^{y_1/y_0} \left(\frac{\sigma_t}{\sigma_{tt}} - 1\right)\frac{\nu}{\nu_t}\, d\left(\frac{y}{y_0}\right) = \frac{\nu}{y_0 k_0^{\frac{1}{2}}}\cdot 2\int_0^{\infty} \left(\frac{\sigma_t}{\sigma_{tt}} - 1\right)\frac{\nu}{\nu_t}\, d\left(\frac{y(\tau_S/\rho)^{\frac{1}{2}}}{\nu}\right). \tag{2.32}$$

Now $y_0 k_0^{\frac{1}{2}}/\nu$ is equal to the constant Y, to which we must later ascribe a value; and the integral is identical with the P function of Spalding & Jayatillaka (1964), which they determined from the examination of experimental data for turbulent flow in smooth pipes to be

$$\int_0^{\infty} \left(\frac{\sigma_t}{\sigma_{tt}} - 1\right)\frac{\nu}{\nu_t}\, d\left(\frac{y(\tau_S/\rho)^{\frac{1}{2}}}{\nu}\right) \equiv P\{\sigma\} \approx 9{\cdot}24\{(\sigma/\sigma_{tt})^{0{\cdot}75} - 1\}, \tag{2.33}$$

where σ_{tt} is best taken as $0{\cdot}9$. Thus the first integral in the square bracket of (2.31) can be evaluated from

$$\int_0^{y_1/y_0} \left(\frac{\sigma_t}{\sigma_{tt}} - 1\right)\frac{\nu}{\nu_t}\, d\left(\frac{y}{y_0}\right) = 2\frac{P}{Y}. \tag{2.34}$$

(iii) The second integral in the square bracket of (2.31) can be split as follows

$$\int_0^{y_1/y_0} \frac{\nu}{\nu_t}\, d\left(\frac{y}{y_0}\right) = \int_0^{1} \frac{\nu}{\nu_t}\, d\left(\frac{y}{y_0}\right) + \int_1^{y_1/y_0} \frac{\nu}{\nu_t}\, d\left(\frac{y}{y_0}\right). \tag{2.35}$$

The first term on the right-hand side can be evaluated for the constant-stress layer. Integration of (2.5) then leads to

$$\int_0^{1} \frac{\nu}{\nu_t}\, d\left(\frac{y}{y_0}\right) = \frac{u_0/(\tau_S/\rho)^{\frac{1}{2}}}{y_0(\tau_S/\rho)^{\frac{1}{2}}/\nu} = 4\,\frac{u_0/k_0^{\frac{1}{2}}}{y_0 k_0^{\frac{1}{2}}/\nu} \equiv 4\frac{U}{Y}, \quad \text{say.} \tag{2.36}$$

Here u_0 is of course the velocity where y equals y_0 in a constant-stress layer; U is thus a constant.

The second term of (2.35) can be evaluated by the use of (2.6) and (2.16), the subscript 1 being omitted. We have

$$\int_1^{y_1/y_0} \frac{\nu}{\nu_t}\, d\left(\frac{y}{y_0}\right) \approx \frac{1}{cY}\left(\frac{2}{1+(3/2m)\,Y'}\right)^{\frac{1}{3}} \int_1^{y_1/y_0} \left(\frac{y}{y_0}\right)^{-(1+3m)} d\left(\frac{y}{y_0}\right). \tag{2.37}$$

Since y_1/y_0 can be taken as much larger than unity, we can write this equation as

$$\int_1^{y_1/y_0} \frac{\nu}{\nu_t}\, d\left(\frac{y}{y_0}\right) \approx \frac{1}{3mcY}\left(\frac{2}{1+(3/2m)\,Y'}\right)^{\frac{1}{3}}. \tag{2.38}$$

(iv) We are now in a position to write down a relation between the Stanton and Reynolds numbers for a separated-flow region; it results from the combination of the following equations: (2.17), (2.30), (2.31), (2.34), (2.36) and (2.38). It is

$$\left(\frac{u_G D}{\nu}\right)^{m/(m+3)} S = \frac{Y^{m/(m+3)}\left(\dfrac{2}{1+(3/2m)Y'}\right)^{1/(m+3)}\left(\dfrac{k_1^{\frac{1}{2}}}{u_G}\right)^{3/(m+3)}\left(\dfrac{D}{y_1}\right)^{m/(m+3)}}{\sigma_{tt}\left[2P+4U+\dfrac{1}{3mc}\left(\dfrac{2}{1+(3/2m)Y'}\right)^{\frac{1}{3}}\right]}. \quad (2.39)$$

Values of the quantities m, c and σ_{tt} have already been recommended; they are 2·0, 0·2 and 0·9 respectively. It is now necessary to estimate values of Y, Y' and U, quantities which characterize conditions at the join of the laminar sublayer and the fully-turbulent region. Here it will be possible to estimate only orders of magnitude.

In the constant-stress layer, 11·6 is often taken as the value of $y(\tau_S/\rho)^{\frac{1}{2}}/\nu$ at the join of the two regions; the linear and the logarithmic velocity profiles, valid respectively for the fully-laminar and fully-turbulent regions, intersect there. Inspection of the data for the turbulent-energy distribution reported by Hinze (1959) shows however that it is only when $y(\tau_S/\rho)^{\frac{1}{2}}/\nu$ equals about 40 that the influence of viscosity vanishes. Since $k_0^{\frac{1}{2}}$ equals $2\tau_S/\rho$ in that case, the value of 80 will be adopted for Y.

Data for the velocity profile in a constant-shear layer show that, where $y(\tau_S/\rho)^{\frac{1}{2}}/\nu$ equals 40, $u(\tau_S/\rho)^{-\frac{1}{2}}$ equals 14·7. It follows that U equals one-half of 14·7 (because $(\tau_S/\rho)^{\frac{1}{2}}$ equals one-half of $k_0^{\frac{1}{2}}$), i.e. 7·35.

The value of Y' in a constant-stress layer varies from zero for $y \gg y_0$ to 2 for $y \ll y_0$. We shall adopt $Y' = 2m/3$, i.e. $\frac{4}{3}$, because it is an intermediate value and because the bracket $2/[1+(3/2m)Y']$ then equals unity. This term enters (2.39) with a small exponent, so little depends on the choice of value for Y'.

Insertion of all the above values into (2.39) yields the equation for the Stanton number of separated flows in the form

$$\left(\frac{u_G D}{\nu}\right)^{0\cdot4} S = \frac{3\cdot2(k_1^{\frac{1}{2}}/u_G)^{0\cdot6}(D/y_1)^{0\cdot4}}{P+15\cdot5}. \quad (2.40)$$

3. Discussion

3.1. *Heat transfer from the downstream half of a circular cylinder*

Richardson (1963) has examined experimental data for the heat transfer from the rear half of a circular cylinder which is held at right angles to a steady stream of air. He recommends an expression which we may write as

$$\left(\frac{u_G D}{\nu}\right)^{0\cdot4} S \approx 0\cdot1\left(\frac{u_G D}{\nu}\right)^{0\cdot067}. \quad (3.1)$$

Here D is the cylinder diameter and u_G is the velocity of the free stream.

In the experiments examined by Richardson, the Reynolds number, $(u_G D/\nu)$ was of the order of 10^5. The right-hand side of the equation is therefore approximately equal to 0·215. We shall now compare this with an estimate of the magnitude of the right-hand side of (2.40).

We may expect $k_1^{\frac{1}{2}}/u_G$ to be of the order of 0·1; measurements of turbulent intensity behind a two-dimensional wall reported by Arie & Rouse (1956), confirm this estimate. As to D/y_1, this may be expected to be of the order of 10. Further, with the Prandtl number σ, equal to 0·7, (2.33) yields $P = -1{\cdot}98$. Insertion of all these results shows the right-hand side of (2.40) to be approximately equal to 0·15. This is to be compared with the 0·215 of the last paragraph.

It must be concluded that the absolute agreement is as satisfactory as could be expected of an analysis based on such uncertain constants. Probably the greatest uncertainty is that concerning D/y_1; it springs from the fact that the model is one-dimensional whereas the experimental situation is two-dimensional.

As to the fact that the present theory suggests that S is proportional to the $-0{\cdot}4$ power of the Reynolds number, whereas Richardson states that the experimental data are best fitted by the $-0{\cdot}333$ power, three things are to be said. First, the scatter of experimental points may be fitted almost as well by curves expressing the former dependency as the latter. Secondly, a different choice of constants a and b in § 2.3 could have led to a different value of m (but not, it must be admitted, to one corresponding to the $-0{\cdot}333$ power; for this is given only by the forbidden condition, $b = 2a/3$). Thirdly, the separated flow at the rear of a circular cylinder is not entirely without shear; therefore, some turbulence is generated near the surface, with the consequence that the exponent is likely to be shifted slightly from that for zero shear (say, $-0{\cdot}4$) towards that for high shear (say, $-0{\cdot}2$).

3.2. *Heat transfer downstream of a step*

Seban *et al.* (1959) have reported heat transfer coefficients measured at the surface of a flat plate downstream of a rearward-facing step, the fluid being air. They state that average values of the heat transfer coefficient for the entire separated-flow region are proportional to the 0·6 power of velocity; this is in precise agreement with the prediction of (2.40), since the Stanton number is proportional to the heat-transfer coefficient divided by the stream velocity.

The absolute agreement between the experimental data and the prediction can be assessed by taking, as an example, the Stanton number at the re-attachment point for a particular experiment, namely, that reported in Seban's figure 4, for a main stream velocity, u_G, of 150 ft./s, and for a step height of 0·81 in. If the reference length D is the step height, insertion of the measured value shows that the value of $(u_G D/\nu)^{0{\cdot}4} S$ was about 0·35.

The value of this quantity which is to be expected depends of course on the values ascribed to $(k_1^{\frac{1}{2}}/u_G)$ and (D/y_G). The former quantity is likely to have the value appropriate to a free mixing layer, because it is fluid from just such a layer which impinges on the re-attachment point; the data of Arie & Rouse (1956) would thus suggest that $(k_1^{\frac{1}{2}}/u_G)$ is of the order of 0·25. With (D/y_1) taken to be 10, as before, and the Prandtl number σ equal to 0·7, we deduce that $(u_G D/\nu)^{0{\cdot}4} S$ should be about 0·26. This value is as close to the experimental value, namely 0·35, as can be required of a theory having so many sources of uncertainty. Once again, the major source results from the fact that turbulent energy is convected to the region of the re-attachment point; a two-dimensional analysis is needed if the relative importance of convection and diffusion is to be determined quantitatively.

3.3. *Heat transfer downstream of a sudden enlargement in a pipe*

Ede *et al.* (1956) measured heat-transfer coefficients downstream of a twofold enlargement in diameter of a pipe through which water was flowing; they found that, in the separated-flow region, the heat-transfer coefficients were three or four times as great as in the region, much farther downstream, where regular pipe flow became re-established. We shall now compare this result with the implications of (2.40) for this case.

According to Spalding & Jayatillaka (1964), the Stanton number for turbulent flow in a smooth pipe, can be calculated from the formula

$$S_{\text{pipe}} = \frac{s^{\frac{1}{2}}}{\sigma_{tt}(P + s^{-\frac{1}{2}})}. \tag{3.2}$$

Here the friction factor s ($\equiv \tau_S/(\rho u_G^2)$, where u_G is the bulk velocity in the pipe), can be calculated from the Blasius formula

$$s \approx 0{\cdot}04\ (u_G D/\nu)^{-\frac{1}{4}}, \tag{3.3}$$

where D is now the pipe diameter.

If the Stanton number for separated flow, given by (2.40), is given the subscript 'sep', combination of that equation with (3.2) and (3.3) yields

$$\frac{S_{\text{sep}}}{S_{\text{pipe}}} \approx 14{\cdot}4 \left(\frac{k_1^{\frac{1}{2}}}{u_G}\right)^{0\cdot6} \left(\frac{D}{y_1}\right)^{0\cdot4} \left(\frac{u_G D}{\nu}\right)^{-0\cdot275} \left\{\frac{P + 5(u_G D/\nu)^{\frac{1}{8}}}{P + 15{\cdot}5}\right\}. \tag{3.4}$$

At $(u_G D/\nu)$ equal to 9200, which happens to make the contents of the curly bracket equal unity, and which also lies well within the range investigated, (3.4) reduces to

$$\frac{S_{\text{sep}}}{S_{\text{pipe}}} \approx 1{\cdot}16 \left(\frac{k_1^{\frac{1}{2}}}{u_G}\right)^{0\cdot6} \left(\frac{D}{y_1}\right)^{0\cdot4}. \tag{3.5}$$

Now $k_1^{\frac{1}{2}}$ is presumably proportional to the velocity of the liquid in the narrow part of the pipe; this is $4u_G$ in the experiments of Ede *et al.* If the ratio is the same as for a mixing layer, namely about 0·25, as cited in the last section, $(k_1^{\frac{1}{2}}/u_G)$ turns out to equal unity. (D/y_1) may be taken as equal to 10 once more, in the absence of further information; then (3.5) reduces to

$$\frac{S_{\text{sep}}}{S_{\text{pipe}}} \approx 2{\cdot}9. \tag{3.6}$$

Once again, a prediction has been obtained which is at least as close to experimental findings as the uncertainty of the foundations permits us to hope.

4. Conclusions

(*a*) The main features of heat transfer in turbulent separated flow appear to be caused by the tendency of turbulent energy, generated in regions of free turbulence, to diffuse towards regions of lower turbulence.

(*b*) The dependence of the heat-transfer rate on the Reynolds number is largely influenced by the relative magnitudes of the constants appearing in the dissipation and the diffusion laws of turbulent energy. The values recommended in §2.3 give good agreement with experiment.

(*c*) The one-dimensional theory of the present paper gives results which agree well, in both tendency and order of magnitude, with the experimental data which have been cited. A two-dimensional theory, which can accommodate also the convection of turbulent energy, must be developed before closer agreement can be expected.

5. Nomenclature

		Equation of first mention
a	Constant in dissipation law.	(2.2)
b	Constant in diffusion law.	(2.3)
c	Constant in total-viscosity law.	(2.6)
D	Reference dimension.	(2.17)
k	Kinetic energy of turbulent motion.	(2.1)
m	Constant.	(2.13)
P	Function of Prandtl number.	(2.33)
p'	Constant expressing pressure gradient.	(2.23)
q	Heat flow rate per unit area.	(2.29)
S	Stanton number.	(2.29)
s	Dimensionless shear stress at wall.	(3.18)
t	Temperature, measured above that of wall.	(2.28)
U	Constant.	(2.36)
u_G	Reference velocity.	(2.17)
u', v', w'	Fluctuating components of velocity in three directions at right angles.	(2.1)
Y	Constant.	(2.16)
Y'	Constant.	(2.15)
y	Distance from wall along normal.	(2.2)
ν	Kinematic viscosity.	(2.5)
ρ	Density.	(2.2)
σ	Prandtl number.	(2.10)
τ	Shear stress.	(2.4)

Subscripts:

0	Join of laminar sublayer and fully turbulent region.
1	Outer boundary of one-dimensional turbulent region.
t	Total, with contributions from both molecular and turbulent transport.
tt	Applied to σ to denote constant value for fully-turbulent conditions.
S	Surface.
pipe	Valid for fully-developed pipe flow.
sep	Valid for separated flow.

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